

# Analysis I (Math 121)

## Resit – Correction

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1. Find a sequence neither decreasing nor increasing that converges to 1. (2 pts.)

**Answer:** Let  $a_n = 1 + \frac{(-1)^n}{n}$ . It is clear that  $\lim_{n \rightarrow \infty} a_n = 1$ . Since the subsequence  $(a_{2n})_n$  is decreasing and converges to 1 and the subsequence  $(a_{2n+1})_n$  is increasing and converges to 1, the sequence  $(a_n)$  is neither increasing nor decreasing.

2. Let  $(a_n)_n$  be a convergent sequence of real numbers. Suppose that  $a_n \in \mathbb{Z}$  for all  $n$ . Is it true that  $\lim_{n \rightarrow \infty} a_n \in \mathbb{Z}$ ? (4 pts.)

**Answer:** Yes, it is true. In fact this is true even for Cauchy sequences: A Cauchy sequence  $(a_n)_n$  whose terms are in  $\mathbb{Z}$  is eventually constant, i.e. there is an  $N$  such that  $a_n = a_N$  for all  $n \geq N$ , and this implies of course that  $\lim_{n \rightarrow \infty} a_n = a_N \in \mathbb{Z}$ . So, let us show that the Cauchy sequence  $(a_n)_n$  is eventually constant.

In the definition of Cauchy sequences, take  $\epsilon = 1/2$ . Thus, there is an  $M$  such that for all  $n, m > M$ ,  $|a_n - a_m| < 1/2$ . But since  $a_n$  and  $a_m$  are in  $\mathbb{Z}$ , this means that for all  $n, m > M$ ,  $|a_n - a_m| = 0$ , i.e. that  $a_n = a_m$ . Now take  $N = M + 1$ .

3. Let  $(q_n)_n$  be a convergent sequence of real numbers. Suppose that  $q_n \in \mathbb{Q}$  for all  $n$ . Is it true that  $\lim_{n \rightarrow \infty} q_n \in \mathbb{Q}$ ? (3 pts.)

**Answer:** Of course not! In fact every real number is the limit of a rational sequence. Indeed, let  $r \in \mathbb{R}$ . Let  $n \in \mathbb{N} \setminus \{0\}$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there is a rational number  $q_n \in (r - 1/n, r)$ . Since  $r - 1/n < q_n < r$ , by the Sandwich Lemma,  $\lim_{n \rightarrow \infty} q_n = r$ .

4. Let  $(a_n)_n$  be a convergent sequence of real numbers. Suppose that  $5a_n/2 \in \mathbb{N}$  for all  $n$ . What can you say about  $\lim_{n \rightarrow \infty} a_n$ ? (4 pts.)

**Answer:** Let  $\lim_{n \rightarrow \infty} a_n = r$ . Then  $\lim_{n \rightarrow \infty} 5a_n/2 = 5r/2$ . By hypothesis and by part 2,  $5r/2 \in \mathbb{Z}$ . Thus  $r = 2n/5$  for some  $n \in \mathbb{N}$ .

5. Let  $(a_n)_n$  be a sequence of real numbers such that the subsequence  $(a_{2n})_n$  converges. Does the sequence  $(a_n)_n$  converge necessarily? (2 pts.)

**Answer:** Of course not! We can have  $a_{2n} = 1/n$  and  $a_{2n+1} = n$ .

6. Let  $(a_n)_n$  be a sequence of real numbers such that the sequence  $(a_n^2)_n$  converges to 1. Does the sequence  $(a_n)_n$  converge necessarily? (2 pts.)

**Answer:** Of course not! We can have  $a_n = (-1)^n$ . Then  $(a_n)_n$  is a sequence of alternating ones and minus ones, so that it diverges. And since  $a_n^2 = 1$ , the sequence  $(a_n^2)_n$  converges to 1.

7. Let  $(a_n)_n$  be a sequence of real numbers such that the subsequences  $(a_{2n})_n$  and  $(a_{2n+1})_n$  both converge. Does the sequence  $(a_n)_n$  converge necessarily? (2 pts.)

**Answer:** Of course not! We can have  $a_n = (-1)^n$ . Then  $(a_n)_n$  is a sequence of alternating ones and minus ones, so that it diverges. And since  $a_{2n} = 1$  and  $a_{2n+1} = -1$ , the sequence  $(a_{2n})_n$  converges to 1 and the sequence  $(a_{2n+1})_n$  converges to  $-1$ .

8. Let  $(a_n)_n$  be a sequence of real numbers such that the sequence  $(a_n^2)_n$  converges to 0. Does the sequence  $(a_n)_n$  converge necessarily? (8 pts.)

**Answer:** Yes! Let  $\epsilon > 0$ . Let  $\nu = \sqrt{\epsilon}$ . Since the sequence  $(a_n^2)_n$  converges to 0, there is an  $N$  such that for all  $n > N$ ,  $|a_n^2| < \nu$ , i.e.  $|a_n|^2 < \epsilon^2$ . Since  $|a_n|$  and  $\nu$  are positive, this implies that  $|a_n| < \epsilon$ . Thus there is an  $N$  such that for all  $n > N$ ,  $|a_n| < \epsilon$ ; i.e. the sequence  $(a_n)_n$  converges to 0.

9. Let  $(a_n)_n$  be a sequence of real numbers such that  $\lim_{n \rightarrow \infty} a_n = \infty$ . Is it true that  $\lim_{n \rightarrow \infty} a_{2n} = \infty$ ? (3 pts.)

**Answer:** Yes! Let  $A$  be any real number.  $\lim_{n \rightarrow \infty} a_n = \infty$ , there is an  $N$  such that for all  $n > N$ ,  $a_n > A$ . Then for  $2n > N$ ,  $a_{2n} > A$ .

10. Assume  $\lim_{n \rightarrow \infty} a_n$  exists and  $a_n \neq 0$  for all  $n$ . Does the sequence  $(a_{2n}/a_{2n+1})_n$  converge necessarily? (5 pts.)

**Answer:** No, the sequence  $(a_{2n}/a_{2n+1})_n$  may not converge if  $\lim_{n \rightarrow \infty} a_n = 0$ . For example, choose

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1/n^2 & \text{if } n \text{ is odd} \end{cases}$$

Clearly  $\lim_{n \rightarrow \infty} a_n = 0$ , but

$$\frac{a_n}{a_{n+1}} = \begin{cases} \frac{(n+1)^2}{n} & \text{if } n \text{ is even} \\ \frac{n+1}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

And the subsequence  $\frac{(n+1)^2}{n}$  diverges to  $\infty$ , although the subsequence  $\frac{n+1}{n^2}$  converges to 0.

On the other hand, if the limit of the sequence  $(a_n)_n$  is nonzero, say  $\ell$ , then the sequence  $(a_{2n}/a_{2n+1})_n$  converges to 1 because  $\lim_{n \rightarrow \infty} a_{2n}/a_{2n+1} = \lim_{n \rightarrow \infty} a_{2n}/\lim_{n \rightarrow \infty} a_{2n+1} = \ell/\ell = 1$ . Note that the last part uses the fact that  $\ell$  is nonzero.

11. Find the following limits and prove your result using **only** the definition. (30 pts.)

a.  $\lim_{n \rightarrow \infty} \frac{3n+105}{5n-79}$

**Answer:**  $\lim_{n \rightarrow \infty} \frac{3n+105}{5n-79} = \frac{3}{5}$ .

**Proof:** Let  $\epsilon > 0$ . Let  $N_1$  be such that  $32 < \epsilon N_1$ . Let  $N = \max(N_1, 395)$ . Now for  $n > N$ , we have,

$$\left| \frac{3n+105}{5n-79} - \frac{3}{5} \right| = \left| \frac{762}{25n-395} \right| = \frac{762}{25n-395} \leq \frac{762}{24n} < \frac{32}{n} < \frac{32}{N_1} < \epsilon.$$

The first equality is simple computation. The second equality follows from the fact  $n > N \geq 395 > 16$  (so that  $25n - 395 > 0$ ). The third inequality follows from the fact that  $n > N \geq 395$ , so that  $25n - 395 \geq 25n - n = 24n$ . The fourth inequality is also a simple computation.

b.  $\lim_{n \rightarrow \infty} \frac{n^2-5n+3}{-100n+2}$

**Answer:**  $\lim_{n \rightarrow \infty} \frac{n^2-5n+3}{-100n+2} = -\infty$ .

**Proof:** It is enough to show that  $\lim_{n \rightarrow \infty} \frac{n^2-5n+3}{100n-2} = \infty$ .

We first note that the two roots of  $n^2 - 5n + 3$  are  $\frac{5 \pm \sqrt{25-12}}{2} = \frac{5 \pm \sqrt{13}}{2}$ , so that if  $n \geq 5 > \frac{9}{2} = \frac{5+\sqrt{16}}{2} > \frac{5+\sqrt{13}}{2}$ , then  $n^2 - 5n + 3 > 0$ .

Now let  $A \in \mathbb{R}$  be any real number. Let  $N = \max(100A + 5, 5)$ . Now, for all  $n > N$ ,

$$\frac{n^2 - 5n + 3}{100n - 2} > \frac{n^2 - 5n + 3}{100n} > \frac{n^2 - 5n}{100n} = \frac{n - 5}{100} > \frac{N - 5}{100} = A.$$

Here, the first inequality follows from the fact that  $n > N \geq 5$ , so that  $n^2 - 5n + 3 > 0$ .

c.  $\lim_{n \rightarrow \infty} \frac{n-8}{2n^3-89}$ .

**Answer:**  $\lim_{n \rightarrow \infty} \frac{n-8}{2n^3-89} = 0$ .

**Proof:** Let  $\epsilon > 0$ . Let  $N = \max(1/\epsilon, 89)$ . Now for  $n > N$ ,  $\left| \frac{n-8}{2n^3-89} \right| = \frac{n-8}{2n^3-89} < \frac{n}{2n^3-n} = \frac{1}{2n^2-1} < \frac{1}{n^2} < \frac{1}{n} < \epsilon$ .

12. Find (16 pts. Justify your answers).

a.  $\lim_{n \rightarrow \infty} \left( \frac{2}{3} + \frac{6n}{n^2+1} \right)^{3n}$ .

**Answer:**  $\lim_{n \rightarrow \infty} \left( \frac{2}{3} + \frac{6n}{n^2+1} \right)^{3n} = 0$ .

**Proof:** We use the fact that  $2/3 < 1$ . Since  $\lim_{n \rightarrow \infty} \frac{6n}{n^2+1} = 0$ , there is an  $N$  such that for all  $n > N$ ,  $\frac{6n}{n^2+1} < 1/6$ . Then  $0 \leq \left( \frac{2}{3} + \frac{6n}{n^2+1} \right)^{3n} = (2/3 + 1/6)^{3n} = (5/6)^{3n}$ . By Sandwich Lemma  $\lim_{n \rightarrow \infty} \left( \frac{2}{3} + \frac{6n}{n^2+1} \right)^{3n} = 0$ .

b.  $\lim_{n \rightarrow \infty} \left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n}$ .

**Answer:**  $\lim_{n \rightarrow \infty} \left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n} = \infty$ .

**Proof:** We use the fact that  $5/4 > 1$ . Since  $\lim_{n \rightarrow \infty} \frac{7}{n^5} = 0$ , there is an  $N$  such that for all  $n > N$ ,  $\frac{7}{n^5} < 1/8$ . Then  $\left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n} > (5/4 - 1/8)^{n^n} = (9/8)^{n^n} \geq (9/8)^n$ . Since  $(9/8) > 1$ ,  $\lim_{n \rightarrow \infty} (9/8)^n = \infty$ . The result follows.

13. Find  $\lim_{n \rightarrow \infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}}$ . (10 pts.)

**Answer:**  $\lim_{n \rightarrow \infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$ .

**Proof:** Since  $\lim_{n \rightarrow \infty} \left(\frac{n^2-1}{n^3-n-5}\right) = 0$ , there is an  $N_1$  such that for all  $n > N_1$ ,  $\frac{n^2-1}{n^3-n-5} < 1/2$ . On the other hand, for  $n > 3$ ,  $\frac{n^2-1}{2n-3} < \frac{n^2-1}{n} < n$ . Let  $N = \max(3, N_1)$ . Now for  $n > N$ ,  $\left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} < (1/2)^{\frac{n^2-1}{2n-3}} < (1/2)^n$ . Since the right hand side converges to 0, by Sandwich Lemma,  $0 \leq \lim_{n \rightarrow \infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$ . (For the first inequality, one needs the fact that  $n^3 - n - 5 > 0$  for  $n \geq 2$ . This follows from the facts that  $2^3 - 2 - 5 = 1 > 0$  and  $n^3 - n - 5 < (n+1)^3 - (n+1) - 5$ . And this last inequality is easy to show).

14. Show that the series  $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3}$  converges. Find an upper bound for the sum. (10 pts.)

**Answer:**  $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3} = \sum_{n=1}^{\infty} (1/n)^{n/3} < \sum_{n=1}^{\infty} 1/2^{n/3} = \sum_{n=1}^{\infty} 1/2^{3n/3} + \sum_{n=0}^{\infty} 1/2^{\frac{3n+1}{3}} + \sum_{n=0}^{\infty} 1/2^{\frac{3n+2}{3}}$   
 $= \sum_{n=1}^{\infty} 1/2^n + \frac{1}{2^{1/3}} \sum_{n=0}^{\infty} 1/2^n + \frac{1}{2^{2/3}} \sum_{n=0}^{\infty} 1/2^n = 1/2 + 2^{-1/3} + 2^{-2/3} < 5$ .

15. Let  $(a_n)_n$  be a sequence of real numbers. Assume that there is an  $r > 1$  such that  $|a_{n+1}| \geq r|a_n|$  for all  $n$ . What can you say about the convergence or the divergence of  $(a_n)_n$ ? (6 pts.)

**Answer:** The sequence diverges. Furthermore the sequence diverges to  $\infty$  if it is eventually positive and to  $-\infty$  if it is eventually negative.

**Proof:** One can show by induction on  $n$  that  $|a_n| > r^n|a_0|$ . Thus  $\lim_{n \rightarrow \infty} |a_n| = \infty$  (because  $r > 1$ ). It should now be clear that the answer is valid.