

**Introduction to Abstract Mathematics**  
**Final Exam**  
**1996-7**

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**Notation:** The letters  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  denote respectively the set of natural numbers, integers, rational numbers and real numbers. Recall that the last three are groups under addition. If  $X$  is a set,  $\wp(X)$  denotes the set of subsets of a set.

**Note:** You should attempt to solve all the questions. An answer with no proof or justification will not be accepted.

Some questions may depend on the previous ones.

**1a.** (5 pts.) Is the set of functions from  $\mathbb{R}$  into  $\mathbb{R} \setminus \{0\}$  a group under the multiplication of functions (If  $f$  and  $g$  are functions from  $\mathbb{R}$  into  $\mathbb{R}$ , the function  $fg$  is defined to be the function whose value at  $x$  is  $f(x)g(x)$ ; i.e.  $(fg)(x) = f(x)g(x)$ )?

**1b.** (5 pts.) Is the set of functions from  $\mathbb{R}$  into the open interval  $(1/2, 2)$  a group under the multiplication of functions?

**1c.** (5 pts.) Is the set of functions from  $\mathbb{R}$  into  $\mathbb{N}$  a group under the multiplication of functions? What about under the addition of functions?

**2a.** (10 pts.) For two sets  $A$  and  $B$ , define  $A \Delta B$  to be  $(A \cup B) \setminus (A \cap B)$ . Show that the set of subsets  $\wp(X)$  of a set  $X$  is a group of exponent 2 under this operation  $\Delta$  (which is called “the symmetric difference”).

**2b.** (10 pts.) Let  $f: X \rightarrow Y$  be any function. Show that the map  $g: \wp(Y) \rightarrow \wp(X)$  defined by  $g(A) = f^{-1}(A)$  is a group homomorphism. (Here,  $\wp(Y)$  and  $\wp(X)$  are considered as groups under  $\Delta$ ).

**2c.** (5 pts.) Show that if there is a bijection between two sets  $X$  and  $Y$ , then the above groups  $\wp(X)$  and  $\wp(Y)$  are isomorphic to each other.

**3a.** (10 pts.) Show that a subgroup of the group  $\mathbb{Z}$  is of the form  $n\mathbb{Z}$  for some integer  $n$ .

**3b.** (5 pts.) Given two integers  $n$  and  $m$ , find  $x$  such that  $n\mathbb{Z} \cap m\mathbb{Z} = x\mathbb{Z}$  as a function of  $n$  and  $m$ .

**4a.** (5 pts.) Let  $G$  be the set of sequences of real numbers that converge to some real number. Show that  $G$  is a group under addition.

**4b.** (5 pts.) Let  $H$  be the set of sequences of real numbers that converge to 0. Show that  $H$  is a subgroup of  $G$ .

**4c.** (5 pts.) Show that the map  $f: G \rightarrow \mathbb{R}$  be defined by  $f((a_n)_n) = \lim_{n \rightarrow \infty} a_n$  is a group homomorphism.

4d. (5 pts.) What is the kernel and the image of  $f$ ?

5. (10 pts.) Prove that  $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ .

6. (16 pts.) Find

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sin(n)/n, \\ & \lim_{n \rightarrow \infty} n \sin(1/n), \\ & \lim_{x \rightarrow 0} \sin(2x)/3x \\ & \lim_{n \rightarrow \infty} 3^{n+1}/2^{2n}. \end{aligned}$$