

**MATH 311 Basic Group Theory**  
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**Problem set 6**

1. Let  $F$  be a finite field,  $|F| = p^n$ . Show that  $\text{UT}_n(F)$  is a Sylow  $p$ -subgroup of  $\text{GL}_n(F)$ .
2. Find Sylow subgroups of the groups  $S_3$  and  $S_4$ .
3. Find all Sylow 2-subgroups of the direct sum of countably many copies of  $S_3$ , and show that there are non-conjugate Sylow 2-subgroups.
4. Prove that any group of order 50 is not simple.
5. Let  $P$  be a Sylow  $p$ -subgroup of a finite group  $G$ , and  $a$  be a  $p$ -element of  $G$ . Prove that if  $P^a = P$  then  $a \in P$ .
6. Prove that if  $P$  is a Sylow  $p$ -subgroup of a finite group  $G$  then  $P$  is a Sylow subgroup of  $N(P)$  and is the only Sylow subgroup of  $N(P)$ .
7. Prove that if  $P$  is a Sylow  $p$ -subgroup of a finite group  $G$  then  $N(N(P)) = N(P)$ .
8. Let  $G$  be a finite  $p$ -group. Prove that  $N(H) \neq H$ , for any proper subgroup  $H$  of  $G$ .
9. Prove that any subgroup of order  $p^{n-1}$  in a group of order  $p^n$  is normal.
10. Let  $G$  be a finite group and  $\phi : G \rightarrow H$  an epimorphism. Prove that
  - (a) if  $P$  is a Sylow  $p$ -subgroup of  $G$  then  $\phi(P)$  is Sylow  $p$ -subgroup of  $H$ ;
  - (b) if  $Q$  is a Sylow  $p$ -subgroup of  $H$  then there is a Sylow  $p$ -subgroup  $P$  of  $G$  such that  $Q = \phi(P)$ .