

Math 211 Algebra  
Final  
January 16, 2006-01-14

A module is called **simple** if it has no nontrivial proper submodules. A module is called **semisimple** if it is a direct sum of simple modules.

1. Classify all simple  $\mathbb{Z}$ -modules. Find a  $\mathbb{Z}$ -module which is not semisimple.
2. Let  $R$  be a ring and  $N$  and  $M$  be two simple  $R$ -modules. Show that any  $R$ -module homomorphism from  $N$  into  $M$  is an isomorphism. Deduce that  $\text{End}_R(M)$  is a division ring.
3. Show that the following three conditions on a module  $M$  are equivalent:
  - a)  $M$  is a direct sum of simple submodules.
  - b)  $M$  is a sum of simple submodules.
  - c) Every submodule  $N$  of  $M$  is a direct summand of  $M$ , that is, there is a submodule  $N'$  such that  $M = N \oplus N'$ .Note: You need Zorn's Lemma.
4. Conclude that submodules and quotients of semisimple modules are semisimple.