Math 111
Final Exam

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1. (5 pts.) What is the intersection of all the subsets of $\mathbb{R}$ whose complement is finite.

2. Let $(X_n)_{n \in \mathbf{N}}$ be a sequence of sets. Define:

$\liminf X_n = \{x : \text{there is an } N \text{ such that } x \in X_n \text{ for all } n > N\}$

$\limsup X_n = \{x : x \in X_n \text{ for infinitely many } n\}$.

2a. (1 pts.) Show that $\liminf X_n \subseteq \limsup X_n$ always.

2b. (2 pts.) Show that for any integer n, the sets $\liminf X_n$ and $\limsup X_n$ are independent of the sets $X_0, X_1, \ldots, X_{n-1}$.

2c. (4 pts.) Show that if $X_n \subseteq Y_n$ for all but finitely many n, then $\liminf X_n \subseteq \liminf Y_n$ and $\limsup X_n \subseteq \limsup Y_n$.

2d. (8 pts.) Prove or disprove

$$\limsup (X_n \cup Yn) = \limsup Y_n \cup \limsup X_n$$
$$\limsup (X_n \cap Yn) = \limsup Y_n \cap \limsup X_n$$
$$\liminf (X_n \cap Yn) = \liminf Y_n \cap \liminf X_n.$$
$$\liminf (X_n \cup Yn) = \liminf Y_n \cup \liminf X_n.$$

2e. (4 pts.) Show that

$$\limsup X_n = \bigcup_n \left(\bigcap_{i=n}^{\infty} X_i \right)$$

and

$$\liminf X_n = \bigcap_n \left(\bigcup_{i=n}^{\infty} X_i \right).$$

2f. (2 pts.) Show that $\liminf X_n^{\mathrm{c}} = (\limsup X_n)^{\mathrm{c}}$ where X^{c} denotes the complement of the set X in some set containing all the sets X_n.

2g. (4 pts.) Define $\lim X_n = X$ iff $\liminf X_n = X = \limsup X_n$. Show that if the sets X_n are decreasing (i.e. $X_{n+1} \subseteq X_n$ for all n) or increasing (i.e. $X_n \subseteq X_{n+1}$ for all n), then $\lim X_n$ exists.

2h. Find the limsup, liminf and lim (if it exists) of the following sequences of sets:

i) (2 pts.) $X_n = \{z \in \mathbb{N} : n < z \leq 2n\}$.

ii) (2 pts.) $X_n = \begin{cases} [0, n] & \text{if } n \text{ is even} \\ [-1/n, 0] & \text{if } n \text{ is odd} \end{cases}$ (these are real intervals).

3. (6 pts.) Does the set $\{\alpha \in \mathrm{Sym}(8) : \alpha^6 = \mathrm{Id}\}$ form a subgroup of Sym(8)? Justify your answer.

4. (12 pts.) Check that the subset $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + y \text{ is even}\}$ is a subgroup of $\mathbb{Z} \times \mathbb{Z}$ and that it is the smallest subgroup containing the elements $(1, 1)$ and $(1, -1)$.

5. (9 pts.) Is the smallest subgroup of (the multiplicative) group $\mathbb{Q}^{>0}$ containg 8 and 12 equal to the smallest subgroup containing 2 and 3? Justify your answer.

6. Let I be a set and G a group. Define,

$\Pi_I\, G$ to be the set of all the functions from I into G and $\oplus_I\, G$ to be the set of functions f from I into G for which $f(i) = 1$ for all but finitely many $i \in I$.

For f and g in $\Pi_I\, G$, define $fg \in \Pi_I\, G$ as follows:

$$(fg)(i) = f(i)g(i) \text{ for all } i \in I.$$

6a. (8 pts.) Show that $\Pi_I\, G$ is a group under this product and that $\oplus_I\, G$ is a normal subgroup of $\Pi_I\, G$.

6b. (4 pts.) Show that if G is abelian, then so is $\Pi_I\, G$.

7. Let G be a group and X a set. Assume that there is a map $\varphi\colon G \times X \to X$ that satisfies the following conditions (we write gx instead of $\varphi(g,x)$ for $(g, x) \in G \times X$):

i) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

ii) $1x = x$ for all $x \in X$.

7a. (7 pts.) Show that the set $R(\varphi) :=\{g \in G\colon gx = x \text{ for all } x \in X\}$ is a normal subgroup of G.

7b. (7 pts.) For $\overline{g} \in G/R(\varphi)$ and $x \in X$, define $\overline{g}\, x = gx$. Show that this definition makes sense (i.e. show that the product $\overline{g}\, x$ is well-defined, i.e. that it depends only on the class $\overline{g}$ of g and not on the choice of g).

7c. (7 pts.) Show that $\overline{\varphi} : G/R(\varphi) \times X \to X$ defined by $\overline{\varphi}(\overline{g}, x) = \overline{g}\, x$ satisfies i and ii.

7d. (6 pts.) Show that $R(\overline{\varphi}) = 1$.