

1. Let G be a group. The equation $(xy)^n = x^n y^n$ holds for all $x, y \in G$ if and only if G is abelian.
2. Describe the symmetry group of an isosceles triangle that is not equilateral.
3. Find a group G and subgroups H and K of G such that $H \trianglelefteq K$ and $K \trianglelefteq G$ and $H \not\trianglelefteq G$.
4. Find all groups which have exactly three subgroups.
5. \mathbb{Q} is not finitely generated.
6. Let G be a group, and let G^G be the set of all functions from G into G . For $\alpha, \beta \in G^G$, define $\alpha + \beta \in G^G$ by declaring $x^{\alpha+\beta} := x^\alpha x^\beta$ for all $x \in G$. Is G^G a ring under this addition and the composition of maps?
7. Let G, G^G and $+$ be as in the previous question. If α and $\beta \in G^G$ are homomorphisms, what is the necessary and sufficient condition for $\alpha + \beta$ to be a homomorphism?
8. Let $C_n = \langle g \rangle$ be a cyclic group of order $n \in \mathbb{N}$. A homomorphism $\varphi : C_n \rightarrow H$, where H is an arbitrary group, is completely determined by $g\varphi$. A homomorphism $\varphi : C_n \rightarrow C_n$ is an automorphism if and only if $o(g\varphi) = n$. Prove that $\text{Aut}(C_n) \cong \mathbb{Z}_n^\times$.
9. Let $S_3 = \langle a, b, c \rangle$ be the symmetric group on 3 letters, $a := (12)$, $b := (13)$, $c := (23)$. A homomorphism $\varphi : S_3 \rightarrow H$, where H is an arbitrary group, is completely determined by $a\varphi, b\varphi$ and $c\varphi$. What are the necessary and sufficient conditions for a homomorphism $\varphi : S_3 \rightarrow S_3$ to be an automorphism? Prove that $\text{Aut}(S_3) \cong S_3$.
10. Prove that $\text{Aut}(V_4) \cong S_3$ and $\text{Aut}(D_8) \cong D_8$.
11. Let $\langle a \rangle$ and $\langle b \rangle$ be cyclic groups of order 2. The free product $F = \langle a \rangle * \langle b \rangle$ is an infinite dihedral group.
12. Suppose that a finite group G acts on a set X , and put $\text{Fix}_X(g) := \{x \in X : xg = x\}$. Then the number of orbits is given by the formula $\frac{1}{|G|} \sum_{g \in G} |\text{Fix}_X(g)|$.
13. If $\sigma = (12)(34) \in S_4$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \in S_4$, what is σ^τ ? Generalize. Describe the conjugacy classes of S_4 .
14. Describe the conjugacy classes of S_5 , of S_n . How many elements are there in each class?
15. Find the conjugacy classes of A_5 and the number of elements in each class. Conclude that A_5 is a simple group.
16. Let p, q, r be distinct prime numbers. If G is a group of order pqr , then G is

not simple.

17. Let G be a finite group, H a subgroup of G . Let P be a Sylow p -subgroup of G .

If $H \trianglelefteq G$, then $H \cap P$ is a Sylow p -subgroup of H . If $H \not\trianglelefteq G$, then $H \cap P$ is not necessarily a Sylow p -subgroup of H . Any Sylow p -subgroup of H has the form $H \cap P'$, where P' is a Sylow p -subgroup of G .