

Algebra II (Math 212)

Final Exam
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1. Let K be an algebraically closed field. Find the number of conjugacy classes of elements of $GL_5(K)$ whose characteristic polynomial is $(T - 2)^5$.

2. Let π be a set of primes. A torsion group is called a π -group if the prime divisors of its elements' orders are in π . Show that every group has a maximal π -subgroup.

3. Let R be a ring with 1 and M an R -module. Let $\varphi : M \rightarrow R$ be an R -module epimorphism. Show that $M \approx \text{Ker}(\varphi) \oplus R$. [Hint: Let $e \in M$ be such that $\varphi(e) = 1$. We have $m = (m - \varphi(m)e) + \varphi(m)e$].

4. Let R be a ring and G a group. Consider the set $R[G]$ of the formal sums

$$\sum_{g \in G} r_g g$$

where $r_g \in R$ for all g and only finitely many of them is nonzero (in other words, by ignoring $0g$'s we may assume that the sum is finite). $R[G]$ becomes a ring and a left R -module with the following operations:

$$\left(\sum_{g \in G} r_g g \right) + \left(\sum_{g \in G} s_g g \right) = \sum_{g \in G} (r_g + s_g) g$$

$$\left(\sum_{g \in G} r_g g \right) \left(\sum_{g \in G} s_g g \right) = \sum_{g \in G} \left(\sum_{hk=g} r_h s_k \right) g$$

$$r \left(\sum_{g \in G} r_g g \right) = \sum_{g \in G} (r r_g) g.$$

4a. If R has n elements and G has m elements, how many elements does $R[G]$ have?

4b. Let R be the ring $\mathbf{Z}/2\mathbf{Z} = \{0, 1\}$ and G be the group $\mathbf{Z}/2\mathbf{Z} = \{1, x\}$ (with $x^2 = 1$). Write the elements of $R[G]$. Find its invertible and nilpotent elements.

4c. Show that $R^* \leq R[G]^*$ and $G \leq R[G]^*$.

Define the map $\varphi : R[G] \rightarrow R$ by the rule

$$\varphi \left(\sum_{g \in G} r_g g \right) = \sum_{g \in G} r_g.$$

4d. Show that φ is a ring and an R -module epimorphism.

4e. Apply Question 2 to $R[G]$ and φ .

4f. Show that, for $g \in G$, $1 - g \notin R[G]^*$.

4g. Show that for $g \in G$, $1 + g \notin \mathbf{Z}[G]^*$.

4h. Let $p \neq 2$ be a prime. Show that if R is a ring of characteristic p and $g \in G$ has order p , then $1 + g \in R[G]^*$.

4i. Let R be a domain and G be the group $\mathbf{Z} = \langle x \rangle$. Show that $R[G]^* = \{rx^n : r \in R^*, n \in \mathbf{Z}\}$.