2008-2009 Master's Program Acceptance Examination

1. Show with very elementary methods that for any prime p and any integer x we have $x^p \equiv x \bmod p$.

2. For a natural number $n > 0$ let $\varphi(n)$ denote the number of positive natural numbers $m \leq n$ which are prime to n. **1a.** Show that if n and m are prime to each other then $\varphi(nm) = \varphi(n)\varphi(m)$. **1b.** Show that $n = \sum_{d|n} \varphi(d)$.

3. Let V be a vector space of finite dimension and U and W be two subspaces of V. Show that $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.

4. **4a.** Let $\mathbf{F}_q$ be a field with q elements. Find the number of elements of $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{SL}_n(\mathbf{F}_q)$. **4b.** Let K be a field. Let $Z = \{$nonzero $n \times n$ scalar matrices over $K\}$. Let $\mathrm{PGL}_n(K) = \mathrm{GL}_n(K)/Z$ and $\mathrm{PSL}_n(K) = \mathrm{SL}_n(K)/(Z \cap \mathrm{SL}_n(K))$. Show that if every element of K has an n^{th} root then $\mathrm{PSL}_n(K) \approx \mathrm{PGL}_n(K)$.

5. For $n, m \in \mathbb{Z}$ prime to each other and $m > 0$ define $f(n/m) = 1/m$. Thus f is a function from $\mathbb{Q}$ into $\mathbb{Q}$. For example $f(0) = f(0/1) = 1/1 = 1$, $f(4) = 1$, $f(2/3) = 1/3$, $f(8/12) = 1/3$. **3a.** Show that f is nowhere continuous. **3b.** Let $a \in \mathbb{Q}$. Let

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Show that g is continuous at a.

6. Show that the function $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not continuous but that it satisfies the conclusion of the intermediate value theorem. Does the function admit a primitive on $\boldsymbol{R}$? What happens if we change the value of f at 0 to any value $\alpha \in \mathbb{R}$?

7. Show that for any continuous and bounded function $f: \mathbb{R} \to \mathbb{R}$ there is an x such that $f(x) = x$.

8. Let X be an infinite set. Call a subset of X open if it is either empty or its complement in X is finite. Show that this defines a non-Hausdorff topology on X. Find all its convergent sequences and their limits.