Galois Theory

Febr. 2001
Final
Ali Nesin

Open books!

1. Prove that the following two properties are equivalent:
(a) Every algebraic extension of K is separable.
(b) Either $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) = p > 0$ and every element of K has a p^{th} root in K.

2. Let E be an algebraic extension of F. Show that every subring of E that contains F is a field.

3. Let $E = F(x)$ where x is transcendental over F.
3a. Let $F < K \leq F(x)$. Show that $F(x)$ is algebraic over K.
3b. Let $y = f(x)/g(x)$ where f and g are prime to each other. Let $n = \max(\deg(f), \deg(g))$. Show that $[F(x) : F(y)] \leq n$.
3c. Everything as above. Prove that $[F(x) : F(y)] = n$.

4. Show that there is a maximal subfield of $\mathbf{C}$ that does not contain $\sqrt{2}$.

5. Let $p_1, p_2, ..., p_n$ be distinct prime numbers. What is the Galois group of the polynomial $(X^2 - p_1) \ldots (X^2 - p_n)$ over $\mathbf{Q}$?

6. Find the Galois group of $X^3 - X + 1$ over $\mathbf{Q}$, $\mathbf{F}_2$ and $\mathbf{F}_7$.

7. Find the roots of unity of the fields $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(i)$, $\mathbf{Q}(\sqrt{-2})$.

8. Let ζ be a primitive n^{th} root of unity. Let $K = \mathbf{Q}(\zeta)$.
8**a.** If $n = p^r$ ($r \geq 1$) is a prime power, show that $N_{K/\mathbf{Q}}(1 - \zeta) = p$.
8b. If n is divisible by at least two distinct primes, show that $N_{K/\mathbf{Q}}(1 - \zeta) = 1$.

9. Let k be a field, K its algebraic closure, $\sigma \in \mathrm{Gal}(K/k)$, $F = \mathrm{Fix}_K(\sigma)$. Show that every finite extension of F is cyclic.